FINAL: INTRODUCTION TO ALGEBRAIC GEOMETRY

Date: **13th May 2013**
The Total points is **110** and the maximum you can score is **100** points.

A **ring** would mean a **commutative ring with identity**.

(1) (5+5+10=20 points) Let $f : A \to B$ be a ring homomorphism. Show that if $P \in \mathrm{Spec}(B)$ then $f^{-1}(P) \in \mathrm{Spec}(A)$. Define Zariski toplogy on $\mathrm{Spec}(A)$. Let $B = A/\sqrt{(0)}$ and f be the quotient map. Show that the map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ sending P to $f^{-1}(P)$ is a homeomorphism with respect to the Zariski topology.

(2) (5+10=15 points) Let X be an algebraic subset of an affine or projective space. When is X called irreducible? Assuming X to be an algebraic subset of affine space, show that X is irreducible iff the coordinate ring of X is a domain.

(3) (5+15=20 points) Let R be a ring and M an R-module. When is M called a projective R-module? For a prime ideal P in R, let M_P denote the localization of M with respect to the multiplicative set $R \setminus P$. Show that M_P is a free R_P-module if M is a projective R-module.

(4) (5+5=10 points) State Going up theorem. Give an example of a ring extension for which "going up" does not hold.

(5) (5+5+5=15 points) Let R be a ring and I an ideal of R. Define minimal primes of R. Compute the irreducible components of the affine algebraic subset of $\mathbb{A}^2_{\mathbb{C}}$ defined by the polynomial $f(x,y) = (x^2+y^2)(x^2+y^2+1)$. What are the minimal primes of the ideal (f) in $\mathbb{C}[x,y]$?

(6) (5+5+5=15 points) Let $f : X \to Y$ be a morphism of irreducible affine varieties. When is f called a dominant morphism? Prove or disprove:
  (a) The natural morphism $i : X \to Y$ where X is a closed subset of Y is a dominant morphism.
  (b) The natural morphism $j : U \to Y$ where U is an affine open subset of Y is a dominant morphism.

(7) (5+10=15 points) Let X be an irreducible projective variety over an algebraically closed field k. What is a rational function on X? Show that the field of rational functions of $\mathbb{P}^n$ is isomorphic to $k(x_1, \dots, x_n)$ where $x_1, \dots, x_n$ are indeterminates (variables).